L^2-Expansion via Iterated Gradients: Ornstein-Uhlenbeck Semigroup and Entropy

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Abstract

We provide an expansion involving the Ornstein-Uhlenbeck semigroup which gives new classes of inequalities. An expansion involving the Gaussian entropy is also obtained, and in turn, it provides a generalizations of the Gaussian logarithmic Sobolev inequality. This last generalization is also reinterpreted in terms of the hypercontractivity of the Ornstein-Uhlenbeck semigroup. The results are also presented on Wiener space.

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1 Introduction

In these notes, we wish to continue some investigations on inequalities started in [HK] and [HP-A]. The main motivation for doing so comes from the works of Beckner [Be], and Ledoux [L].

Before going any further, let us briefly recall some results. Let X be a multivariate standard normal random variable, $X \sim N(0, I)$, and let $f : \mathbf{R}^N \longrightarrow \mathbf{R}$ be a smooth function, with square integrable iterated gradients, then for any $n \geq 1$,

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} E|\nabla^k f(X)|^2 \leq Var f(X) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} E|\nabla^k f(X)|^2, \tag{1.1}$$

where the ∇^k are the iterated gradients, and where $|\cdot|$ is the Euclidian norm in the corresponding sapces $\mathbf{R}^{kN}$. For $N = 1$, these inequalities were obtained in [HK], while for arbitrary N, they were obtained in [HP-A] as an immediate consequence of a general covariance identity for Wiener functionals. Following a semigroup approach, Ledoux [L] (see also Hu [H]) recently obtained identities from which the inequalities (1.1) are also immediate. Furthemore, the results of [H] are given for diffusions, while [L] obtained identities and inequalities in great generality for the generator of Markovian semigroups. We now recall, some elements of the semigroup approach (in the Gaussian case) which will also permit to introduce some of the results in [Be].

Let $\{P_s\}_{s\geq 0}$ be the Ornstein-Uhlenbeck semigroup, then P_s has an integral representation given by

$$P_s f(x) = \int_{\mathbf{R}^N} f(e^{-s}x + (1 - e^{-2s})^{\frac{1}{2}} t) d\gamma_N(t), \quad x \in \mathbf{R}^N, \quad s \geq 0, \tag{1.2}$$

where $d\gamma_N$ is the Gaussian measure with density $(2\pi)^{-\frac{N}{2}} \exp(-\frac{|t|^2}{2})$ and where $f : \mathbf{R}^N \longrightarrow \mathbf{R}$ is integrable with respect to γ_N. Moreover, $\{P_s\} = e^{-sL}$, where the generator L acts on smooth functions f via $Lf(x) = -\Delta f(x) + x \cdot \nabla f(x)$, where Δ is the usual Laplacian. With these notions, it is shown in [Be] that for $1 \leq p \leq 2$ and $e^{-s} = \sqrt{p-1}$, then

$$E|f(X)|^2 - E|P_s f(X)|^2 \leq (2 - p) E|\nabla f(X)|^2. \tag{1.3}$$

Now, as $s \to +\infty$, or equivalently, as $p \to 1$, then $P_s f \to Ef$ and (1.3) recovers the right hand side of (1.1), for $n = 1$. Hence, it is very natural to wonder if (1.3) (which we were unaware of when [HK] and [HP-A] were written) admits a version with iterated gradients which will also fully recover (1.1). Such a version is possible and obtained in the next section. In the same section, and via the semigroup approach, we also present an identity for $E|P_s f(X)|^2$ from which these new inequalities are also immediate. To finish the section, identities and inequalities are obtained on Wiener space. In Section 3, borrowing the framework of [L], results for the entropy are obtained from which Gross logarithmic Sobolev inequality [G], as well as Nelson hypercontractivity estimate [N], are generalized (see Bakry [Ba] for an exposition of these topics).

2 Ornstein-Uhlenbeck Results

To start we follow the method of [Be], and prove:

Theorem 2.1 *Let $X \sim N(0, I)$, let $1 \le p \le 2$, and let $e^{-s} = \sqrt{p-1}$. Let f be $2n-1$ (resp. $2n$) times differentiable on $\mathbf{R}^N$ and such that $E|\nabla^k f(X)|^2 < +\infty$, $k = 0, 1, \ldots, 2n+1$ (resp. $k = 0, 1, \ldots, 2n$), $n \ge 0$. Then, the left (resp. right) inequality holds:*

$$\sum_{k=0}^{2n+1} (2-p)^k \frac{(-1)^k}{k!} E|\nabla^k f(X)|^2 \le E|P_s f(X)|^2 \le \sum_{k=0}^{2n} (2-p)^k \frac{(-1)^k}{k!} E|\nabla^k f(X)|^2. \quad (2.1)$$

Proof It is enough to check these inequalities using the Hermite expansion of $f(X)$. In fact, by linearity we just need to verify the inequalities on the mutivariate Hermite polynomials $H_\alpha(x) = \prod H_{\alpha_j}(x_j)$ of, say, length $m = \sum \alpha_j$, where the $H_{\alpha_j}(x_j)$ are univariate Hermite polynomials with the normalization

$$e^{-\frac{t^2}{2}+ty} = \sum_{j=0}^{+\infty} \frac{t^j}{j!} H_j(y).$$

But, then the left inequality corresponds to

$$E|\pi_m f(X)|^2 - (p-1)^m E|\pi_m f(X)|^2 \le -\sum_{k=1}^{2n+1} (2-p)^k \frac{(-1)^k m \cdots (m-k+1)}{k!} E|\pi_m f(X)|^2,$$

where $\pi_m f$ is the projection of f onto the linear span of the H_α of length m. In turn, to prove this inequality, it is enough to show that

$$1 - (p-1)^m \le \sum_{k=1}^{2n+1} (2-p)^k \frac{(-1)^{k+1} m(m-1) \cdots (m-k+1)}{k!}.$$

This last estimate now follows from the inequality

$$(1-\theta)^m \geq \sum_{k=0}^{2n+1} \begin{pmatrix} m \\ k \end{pmatrix} (-\theta)^k,$$

which is valid for all $m \geq 0$, setting $\begin{pmatrix} m \\ k \end{pmatrix} = 0$, when $m < k \leq 2n+1$, and also $1-p = \theta - 1$, $0 \leq \theta \leq 1$. The right hand side inequalities similarly follow from

$$(1-\theta)^m \leq \sum_{k=0}^{2n} \begin{pmatrix} m \\ k \end{pmatrix} (-\theta)^k,$$

which is valid for all $m \geq 0$, where $\begin{pmatrix} m \\ k \end{pmatrix} = 0$ when $m < k \leq 2n$, and again $1-p = \theta-1$, $0 \leq \theta \leq 1$.

Remark 2.2 Again, for $p = 1$, (2.1) recovers (1.1) since $\lim_{s\to+\infty} P_s f = Ef$, while the left hand side of (2.1) for $n = 0$ gives (1.3). As in [HK],

$$E|P_s f(X)|^2 = \sum_{k=0}^{+\infty} (2-p)^k \frac{(-1)^k}{k!} E|\nabla^k f(X)|^2$$

if and only if $\lim_{k\to+\infty} \frac{(2-p)^k}{k!} E|\nabla^k f(X)|^2 = 0$. Following [Be], Nelson's hypercontractivity estimate combined with the left hand side of (2.1) gives,

$$\sum_{k=0}^{2n+1} (2-p)^k \frac{(-1)^k}{k!} E|\nabla^k f(X)|^2 \leq (E|f(X)|^p)^{\frac{2}{p}}, \quad n \geq 0, \tag{2.2}$$

while substracting $E|f(X)|^2$ from (2.2), dividing the result by $2-p$ and letting $p \to 2$ gives Gross logarithmic Sobolev inequality. Still following [Be] we mention that the inequalities (2.1) also provide sharpened forms of the Uncertainty Principle (although rather delicate to exploit). In fact, the inequalities (2.1) can be seen as a complement to the L^2-version of the contractive estimate for P_s, while (2.2) measures the deviation from $E|f(X)|^2$ which, of course, always dominates the right hand side of (2.2).

We now follow [L], use a semigroup approach and provide an identity from which (2.1) can be deduced. We will make use of the following two properties of the Ornstein-Uhlenbeck semigroup: The first one comes from the form of the generator, while the second is immediate from the representation (1.2)

$$Ef(X)Lg(X) = E\nabla f(X)\cdot\nabla g(X) \tag{2.3}$$

$$\nabla P_s f = e^{-s}P_s\nabla f \tag{2.4}$$

Also, below, $E|f|$ is short for $E|f(X)|$, while letting $s \to +\infty$, recovers a result of [L],

Theorem 2.3 *Let $X \sim N(0,I)$, let $f : \mathbf{R}^N \longrightarrow \mathbf{R}$ be of class C^{n+1}, $n \geq 0$, and such that $E|\nabla^k f(X)|^2 < +\infty$, $k = 0,1,\dots,n+1$. Then,*

$$E|P_s f|^2 = \sum_{k=0}^{n}(1-e^{-2s})^k\frac{(-1)^k}{k!}E|\nabla^k f|^2 - \frac{(-1)^n}{n!}\int_0^s 2e^{-2t}(e^{-2t}-e^{-2s})^n E|P_t\nabla^{n+1}f|^2dt. \tag{2.5}$$

Proof The proof is done by induction, and (2.5) is first proved for $n = 0$, proceeding as in [L], (using (2.3) and (2.4)).

$$\begin{aligned} E|P_s f|^2 &= E|f|^2 + \int_0^s \frac{d}{dt}E|P_t f|^2 dt \\ &= E|f|^2 - 2\int_0^s EP_t fLP_t f dt \\ &= E|f|^2 - 2\int_0^s E|\nabla P_t f|^2 dt \\ &= E|f|^2 - 2\int_0^s e^{-2t}E|P_t\nabla f|^2 dt, \end{aligned} \tag{2.6}$$

which gives (2.5) for $n = 0$. Now, let us assume that that (2.5) is true for the integer $n \geq 1$. An integration by parts in the integral there gives

$$\begin{aligned} E|P_s f|^2 &= \sum_{k=0}^{n}(1-e^{-2s})^k\frac{(-1)^k}{k!}E|\nabla^k f|^2 + \frac{(-1)^n}{n!}\left[\frac{1}{n+1}(e^{-2t}-e^{-2s})^{n+1}E|P_t\nabla^{n+1}f|^2\right]_0^s \\ &\quad + \frac{(-1)^{n+1}}{(n+1)!}\int_0^s (e^{-2t}-e^{-2s})^{n+1}\frac{d}{dt}E|P_t\nabla^{n+1}f|^2 dt \qquad (2.7) \\ &= \sum_{k=0}^{n+1}\frac{(e^{-2s}-1)^k}{k!}E|\nabla^k f|^2 - \frac{(-1)^{n+1}}{(n+1)!}\int_0^s 2e^{-2t}(e^{-2t}-e^{-2s})^{n+1}E|P_t\nabla^{n+2}f|^2 dt, \end{aligned}$$

where to pass from (2.7) to the next expression, one proceeds as in the proof of (2.6).

It is clear that (2.5) immediately recovers (1.1) and (2.1) and also gives new inequalities (thanks to the alternating sign in the integral in (2.5)). Again, $E|P_s f|^2 = \sum_{k=0}^{\infty}(1-e^{-2s})^k\frac{(-1)^k}{k!}E|\nabla^k f|^2$ if and only if $\lim_{k\to+\infty}\frac{(1-e^{-2s})^k}{k!}E|\nabla^k f(X)|^2 = 0$.

To continue this section, we prove a version of Theorem 2.2 on Wiener's space thus extending the identity (via the parallelogram law) obtained in [HP-A] (we refer to the references of [HP-A] for extensive background material on the anticipative calculus).

Briefly, if F and G are two square integrable Wiener functionals, let $\mathbf{D}^n, n \geq 1$, be the corresponding Sobolev spaces with inner product given by

$$(F, G) = EFG + \sum_{k=1}^{n} E < \nabla^k F, \nabla^k G >_k,$$

where

$$< \nabla^k F, \nabla^k G >_k = \int_0^1 \cdots \int_0^1 \nabla^k_{t_1 \cdots t_k} F \nabla^k_{t_1 \cdots t_k} G dt_1 \cdots dt_k.$$

Let $\{P_s\}_{s\geq 0}$ be the Ornstein-Ulhenbeck operator with generator L. If $F = \sum_{m=0}^{+\infty} I_m(f_m)$, is the chaos expansion of the Wiener functional F, then $P_s F = \sum_{m=0}^{+\infty} e^{-ms} I_m(f_m)$ and $LF = \sum_{m=1}^{+\infty} m I_m(f_m)$. Hence, the corresponding versions of (2.3) and (2.4) are:

$$EFLG = E < \nabla F, \nabla G >_1, \tag{2.8}$$

$$\nabla P_s F = e^{-s} P_s \nabla F. \tag{2.9}$$

Theorem 2.4 *Let $F \in \mathbf{D}^{n+1}$, then*

$$E|P_s F|^2 = \sum_{k=0}^{n} (1-e^{-2s})^k \frac{(-1)^k}{k!} E||\nabla^k F||_k^2 - \frac{(-1)^n}{n!} \int_0^s 2e^{-2t}(e^{-2t}-e^{-2s})^n E||P_t \nabla^{n+1} F||_{n+1}^2 dt. \tag{2.10}$$

Proof The proof closely follows the proof of (2.5). For $n = 0$, using (2.8) and (2.9),

$$\begin{aligned} E|P_s F|^2 &= E|F|^2 + \int_0^s \frac{d}{dt} E|P_t F|^2 dt \\ &= E|F|^2 - 2\int_0^s EP_t F L P_t F dt \\ &= E|F|^2 - 2\int_0^s E||\nabla P_t F||_1^2 dt \\ &= E|F|^2 - 2\int_0^s e^{-2t} E||P_t \nabla F||_1^2 dt. \end{aligned} \tag{2.11}$$

Assuming the result for n, to prove it for $n + 1$, it is enough to integrates by parts as in (2.7) and to use the fact that $\frac{d}{dt} E||P_t \nabla^{n+1} F||_{n+1}^2 = -2e^{-2t} E||P_t \nabla^{n+2} F||_{n+2}^2$, which follows as in the proof of (2.11), but using (2.8) and (2.9) instead of (2.3) and (2.4).

The remainder term in the identity (2.10) looks somehow different from the remainder in the identity obtained in [HP-A] (when $s = +\infty$). Of course, the two remainder are identical. This is seen now in the simplest case, i.e., for n = 0, the general case being done similarly by induction. ¿From (2.11), and using the identities (see [NZ]) $(I+L)^{-1} = \int_0^{+\infty} e^{-t} P_t dt$, $(I+L)^{-1}\nabla = \nabla L^{-1}$ as well as the duality relation for the anticipative integral and the gradient, we have:

$$\begin{aligned} 2\int_0^{+\infty} e^{-2t} E||P_t\nabla F||_1^2 dt &= E\int_0^1\int_0^{+\infty} 2e^{-2t} P_{2t}\nabla_s F dt \cdot \nabla_s F ds \\ &= E\int_0^1 (I+L)^{-1}\nabla_s F\cdot\nabla_s F ds \\ &= E\int_0^1 (\nabla L^{-1})_s(F-EF)\cdot\nabla_s(F-EF)ds \\ &= E\int_0^1 (\nabla L^{-1})_s(F-EF)dW_s\cdot(F-EF) \\ &= ELL^{-1}(F-EF)\cdot(F-EF), \end{aligned}$$

which by the Clark-Ocone formula gives the remainder of [HP-A].

Of course, the alternating sign in (2.10) gives coresponding inequalities. Proceeding as in Remark 2.2, gives the following logarithmic Sobolev inequality on Wiener space (the hypercontractivity of the Ornstein-Uhlenbeck operator on Wiener space is also true (see Watanabe [W]), and in fact, a stronger related result will be proved in the next section)

$$E|F|^2\log|F| - \frac{1}{2}E|F|^2\log E|F|^2 \le \int_0^1 E|\nabla_t F|^2 dt.$$

To finish this section, we take up the approach of [H] who also recovered (1.1). Again let us briefly recall the background there. Let

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x,\ 0\le t\le 1,$$

be a one dimensional stochastic differential equation and let $Q_t(x,y)$ be the density (when it exists) of the associated semigroup. Let us also set

$$Q_s f(x) = \int_{\mathbf{R}} Q_s(x,y)f(y)dy,$$

for integrable f. Then, since $X_0 = x$ and since by Itô's formula

$$\frac{d}{dt}E|Q_{1-t}f(X_t)|^2 = E|\nabla Q_{1-t}f(X_t)|^2 \tag{2.12}$$

we have

$$\begin{aligned} E|Q_{1-s}f(X_s)|^2 &= E|Q_1 f(X_0)|^2 + \int_0^s \frac{d}{dt} E|Q_{1-t}f(X_t)|^2 dt \\ &= |Q_1 f(x)|^2 + \int_0^s E|\nabla Q_{1-t} f(X_t)|^2 dt \qquad (2.13) \\ &= |Q_1 f(x)|^2 + \Big[tE|\nabla Q_{1-t}f(X_t)|^2\Big]_0^s - \int_0^s t\frac{d}{dt}E|\nabla Q_{1-t}f(X_t)|^2 dt \\ &= |Q_1 f(x)|^2 + sE|\nabla Q_{1-s}f(X_s)|^2 - \int_0^s tE|\nabla^2 Q_{1-t}f(X_t)|^2 dt. \qquad (2.14) \end{aligned}$$

Now, iterating (2.14) gives (for appropriate f)

$$\begin{aligned} E|Q_{1-s}f(X_s)|^2 &= |Q_1 f(x)|^2 + \sum_{k=1}^n \frac{(-s)^{k+1}}{k!} E|\nabla^k Q_{1-s}f(X_s)|^2 \\ &\quad - \frac{(-1)^{n+1}}{n!}\int_0^s t^n E|\nabla^{n+1} Q_{1-t}f(X_t)|^2 dt. \qquad (2.15) \end{aligned}$$

Again, the alternating sign in (2.15) gives corresponding inequalities, while putting $s = 1$ in (2.15) and since $Q_0 = I$ recovers a result of [H].

In order to extend the results of [L] to Markovian semigroups (as above) rather than to their generator, a crucial "commutation" property (à la (2.4)) between the semigroup and the gradient is needed. For example, on the sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$, if $P_r,\ 0 \le r < 1$, is the Poisson kernel (see [Be]),

$$P_r \nabla f = r \nabla P_r f. \qquad (2.16)$$

Hence the previous results valid for the Gaussian measure extend to the normalized surface measure on $\mathbf{S}^n$, thus also extending the corresponding results of [Be]. Similar results also hold on the two-point space (see [Ba]) as well as on the infinite dimensional Poisson space defined via chaos expansion, thus recovering in this last case the results of [HP-A]. For general Markovian semigroups, such a commutation does not hold. As indicated to us by M. Ledoux, the best we can be hoped for, is an inequality (in Euclidian norm) when, for example, $\Gamma_2 \ge 0$ (see next section for Γ_2). It is nevertheless possible to obtain an expansion (much less pleasant) by proceeding as in (2.13-2.15). This is not done here. We rather stick to the Gaussian framework to present results on the entropy.

3 Entropy Results

In order to develop the entropy $Ef\log f - Ef\log Ef, f > 0$, the iterated gradients are not rich enough and a finer structure introduced by Ledoux [L] is needed. This structure is formed by multilinear forms, which generalize the "carré du champ" operator $\Gamma_1 = \Gamma$, and its iterates Γ_n. We now recall and adopt some elements of [L] to provide in the Gaussian case an expansion which will recover the expansion for the entropy obtained there.

Briefly, let L be the generator of a Markovian semigroup, and let its domain be contained in a "good" algebra $\mathcal{A}$ of real valued smooth fuctions defined on, say, $\mathbf{R}^N$. The "iterated gradients" Γ_n are the symmetric bilinear forms (introduced by Bakry) from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ defined via

$$\Gamma_0(f,g) = fg,$$

$$2\Gamma_n(f,g) = L\Gamma_{n-1}(f,g) - \Gamma_{n-1}(f,Lg) - \Gamma_{n-1}(g,Lf),\ n \geq 1.$$

In the case of interest to us, i.e., when L is the Ornstein-Uhlenbeck generator (extended to $\mathcal{A}^k$ via $L(f_1,\ldots,f_k) = (Lf_1,\ldots,Lf_k)$ and using also the notations $L(f,f) = Lf, \Gamma_n(f,f) = \Gamma_n f, f \in \mathcal{A}$) it follows that:

$$\Gamma f = |\nabla f|^2,$$

$$\Gamma_n f = \sum_{k=1}^{n} \alpha_k^n |\nabla^k f|^2,$$

where $|\cdot|$ is the Euclidian norm in the appropriate space $\mathbf{R}^{kN}$, and where the α_k^n satisfy some induction relations. The above relations can also be read backward to give

$$Q_n(\Gamma)f = \sum_{k=1}^{n} \frac{1}{k!} Q_n^{(k)}(0)\Gamma_k f = \sum_{k=1}^{n} A_k^n |\nabla^k f|^2. \tag{3.1}$$

For the Ornstein-Uhlenbeck generator, the A_k^n above are all zero unless $k = n$, and

$$Q_n(\Gamma)(f,g) = \nabla^n f \cdot \nabla^n g. \tag{3.2}$$

Thus, (3.2) explains the presence of the iterated gradients in the expansion of the variance, while in general, the $Q_n(\Gamma)$ appear. To develop the entropy, the Γ_n are insufficient and a sequences of multilinear forms $\widetilde{\Gamma}_n$ defined on a subalgebra $\mathcal{L}$ of the Lie algebra $\mathcal{M}(\mathcal{A})$ generated by L, Γ, and their Lie brackets, is needed. More precisely, let

$$\widetilde{\Gamma}_1\ (= \widetilde{\Gamma}) = \Gamma_1 = \Gamma,$$

$$\widetilde{\Gamma}_n = [L + \Gamma, \widetilde{\Gamma}_{n-1}] = [L, \widetilde{\Gamma}_{n-1}] + [\Gamma, \widetilde{\Gamma}_{n-1}],\ n \geq 2,$$

where for $A, B \in \mathcal{L}$, $[A, B] = AB - BA$. In particular, $\widetilde{\Gamma}_2 = \Gamma_2$ and as in (3.1), the "polynomials"

$$Q_n(\widetilde{\Gamma})f = \sum_{k=1}^{n} \frac{1}{k!} Q_n^{(k)}(0)\widetilde{\Gamma}_k f, \ n \geq 1, \tag{3.3}$$

are obtained with again the notation $\widetilde{\Gamma}_k f = \widetilde{\Gamma}_k(f, \cdots, f)$. In this context, and even when L is the Ornstein-Uhlenbeck generator, the computation of the $Q_n(\widetilde{\Gamma})f$ is delicate and nothing like (3.1) seems available (the first few $Q_n(\widetilde{\Gamma})$ are computed in [L]). We will however note two properties in the Gaussian case (for general semigroup, n below should be replaced by λ_n) which will play an important rôle in developing the entropy.

$$Q_{n+1}(\widetilde{\Gamma}) = -nQ_n(\widetilde{\Gamma}) + [L + \Gamma, Q_n(\widetilde{\Gamma})], \tag{3.4}$$

$$\frac{d}{dt} EP_t f Q_n(\widetilde{\Gamma}) \log P_t f = -2nEP_t f Q_n(\widetilde{\Gamma}) \log P_t f - 2EP_t f Q_{n+1}(\widetilde{\Gamma}) \log P_t f. \tag{3.5}$$

After these long preliminaries extracted from [L] and if $\mathcal{S}$ below denote the Schwartz class of C^∞ rapidly decreasing functions we have:

Theorem 3.1 *Let $X \sim N(0, I)$, and let $f > 0$, $f \in \mathcal{S}$, be bounded. Then,*

$$\begin{aligned} 2Ef \log f \quad - \quad 2EP_s f \log P_s f &= \sum_{k=1}^{n} (1 - e^{-2s})^k \frac{(-1)^{k+1}}{k!} Ef Q_k(\widetilde{\Gamma}) \log f \\ + \quad \frac{(-1)^n}{n!} &\int_0^s 2e^{2nt}(e^{-2t} - e^{-2s})^n EP_t f Q_{n+1}(\widetilde{\Gamma}) \log P_t f dt. \end{aligned} \tag{3.6}$$

Proof The proof is done by induction, and (3.6) is first proved for $n = 1$, proceeding as in the proof of Theorem 2.3, using (2.3) and (2.4), and since $ELf = 0$,

$$\begin{aligned} 2EP_s f \log P_s f &= 2Ef \log f + 2 \int_0^s \frac{d}{dt} EP_t f \log P_t f dt \\ &= 2Ef \log f - 2 \int_0^s ELP_t f \log P_t f dt \\ &= 2Ef \log f - 2 \int_0^s E\nabla P_t f \cdot \nabla \log P_t f dt \\ &= 2Ef \log f - 2 \int_0^s e^{-2t} E(P_t f)^{-1} |P_t \nabla f|^2 dt. \end{aligned} \tag{3.7}$$

Integrating by parts, the integral in (3.7) is equal to:

$$\begin{aligned} &\left[-\frac{1}{2} e^{-2t} E(P_t f)^{-1} |P_t \nabla f|^2 \right]_0^s + \frac{1}{2} \int_0^s e^{-2t} \frac{d}{dt} E(P_t f)^{-1} |P_t \nabla f|^2 dt \\ = -\frac{1}{2} e^{-2s} E(P_s f)^{-1} |P_s \nabla f|^2 &+ \frac{1}{2} Ef^{-1} |\nabla f|^2 + \frac{1}{2} \int_0^s e^{-2t} \frac{d}{dt} E(P_t f)^{-1} |P_t \nabla f|^2 dt. \end{aligned} \tag{3.8}$$

Since,

$$\int_0^s \frac{d}{dt} E(P_t f)^{-1} |P_t \nabla f|^2 dt = E(P_s f)^{-1} |P_s \nabla f|^2 - E f^{-1} |\nabla f|^2, \tag{3.9}$$

combining (3.7), (3.8) and (3.9), we get

$$2Ef \log f - 2EP_s f \log P_s f = (1 - e^{-2s}) E f^{-1} |\nabla f|^2 + \int_0^s (e^{-2t} - e^{-2s}) \frac{d}{dt} E(P_t f)^{-1} |P_t \nabla f|^2. \tag{3.10}$$

But,

$$E f^{-1} |\nabla f|^2 = Ef |\nabla \log f|^2 = Ef \widetilde{\Gamma} \log f. \tag{3.11}$$

Hence, using (3.11), (2.3) and (3.5) for $n = 1$, we get

$$\begin{aligned} \frac{d}{dt} E(P_t f)^{-1} |P_t \nabla f|^2 &= \frac{d}{dt} E e^{2t} P_t f \widetilde{\Gamma} \log P_t f \\ &= 2e^{2t} E P_t f \widetilde{\Gamma} \log P_t f - 2e^{2t} E P_t f \widetilde{\Gamma} \log P_t f - 2e^{2t} E P_t f Q_2(\widetilde{\Gamma}) \log P_t f \\ &= -2e^{2t} E P_t f Q_2(\widetilde{\Gamma}) \log P_t f \end{aligned} \tag{3.12}$$

Finally, combining (3.12) with (3.10) gives (3.6) for $n = 1$. Let us now assume that (3.6) is true for the integer n. Integrating by parts, the integral there is equal to

$$\begin{aligned} & - \left[\frac{1}{n+1} (e^{-2t} - e^{-2s})^{n+1} e^{2(n+1)t} E P_t f Q_{n+1}(\widetilde{\Gamma}) \log P_t f \right]_0^s \\ & + \frac{1}{n+1} \int_0^s (e^{-2t} - e^{-2s})^{n+1} \frac{d}{dt} e^{2(n+1)t} E P_t f Q_{n+1}(\widetilde{\Gamma}) \log P_t f dt \end{aligned} \tag{3.13}$$

$$\begin{aligned} = \ & \frac{(1 - e^{-2s})^{n+1}}{n+1} E f Q_{n+1}(\widetilde{\Gamma}) \log f \\ & - \frac{1}{(n+1)} \int_0^s 2e^{-2t} (e^{-2t} - e^{-2s})^{n+1} e^{2(n+2)t} E P_t f Q_{n+2}(\widetilde{\Gamma}) \log P_t f dt, \end{aligned}$$

where to pass from (3.13) to the next expression, one proceeds as in the proof of (3.12) and uses (3.5). The proof is complete.

As already mentioned in [L] (for $s = +\infty$), the expansion (3,6) does not provide corresponding odd and even inequalities à la (2.1). In fact, for different classes of functions, even sums can majorize or minorize $Ef \log f - EP_s f \log P_s f$. Such is also the case of odd sums, except for $n = 1$ since we then have:

Corollary 3.2 Let $X \sim N(0, I)$, and let f be differentiable on $\mathbf{R}^N$ with $E|\nabla f(X)|^2 < +\infty$. Then,

$$Ef^2 \log f^2 - EP_s f^2 \log P_s f^2 \le 2(1 - e^{-2s}) E |\nabla f|^2, \quad s \ge 0. \tag{3.14}$$

Proof For, $n = 1$, and since the remainder in (3.6) is non positive (this can be seen from (3.10) or (3.12)), Theorem 3.1 gives ($f > 0$):

$$2Ef\log f - 2EP_sf\log P_sf \leq (1-e^{-2s})Ef^{-1}|\nabla f|^2. \tag{3.15}$$

Replacing, above, f by f^2 gives the result for positive bounded functions in the Schwartz class. An approximation argument as in [Ba], finishes the proof.

Remark 3.3 (i) The left hand side of (3.14) is non negative since the function $x\log x$ is convex and since P_s is an order preserving contractive semigroup. Of course, letting $s \to +\infty$, recovers Gross logarithmic Sobolev inequality [G]. Furthermore,

$$Ef^2\log f^2 - Ef^2\log Ef^2 \leq \frac{Ef^2\log f^2 - EP_sf^2\log P_sf^2}{(1-e^{-2s})} \leq 2E|\nabla f|^2, \tag{3.16}$$

and the middle quantity in (3.16) is a smooth nonincreasing interpolator (to prove this claim, one can procced as in the proof of (3.6), for $n = 1$ given above) recovering the left quantity at $s = +\infty$ and the right one at $s = 0$.

(ii) Proceeding as in [Ba, Proposition 3.10], i.e., applying (3.15) to P_tf and then solving the resulting delayed differential inequality for the function $\phi(t) = EP_tf\log P_tf$, (3.14) implies: ($f > 0$, say,):

$$EP_tf\log P_tf \leq e^{\frac{-2t}{1-e^{-2s}}}Ef\log f + \frac{2}{1-e^{-2s}}\int_0^t e^{\frac{-2(t-u)}{1-e^{-2s}}}EP_{u+s}f\log P_{u+s}f du, t \geq 0, s > 0. \tag{3.17}$$

Conversely, differentiating (3.17) at $t = 0$ recovers (3.15). Thus (3.14) and (3.17) are equivalent. Similarly, (1.1) itself is equivalent to

$$E|P_tf|^2 \leq e^{\frac{-2t}{1-e^{-2s}}}E|f|^2 + \frac{2}{1-e^{-2s}}\int_0^t e^{\frac{-2(t-u)}{1-e^{-2s}}}E|P_{u+s}f|^2du, \quad t \geq 0, s > 0.$$

In particular, letting $s \to +\infty$ above, we get (once more in the Gaussian case) Proposition 3.10 and 2.3 of [Ba].

(iii) Still following [Ba, Proposition 3.1] and for any $p \neq 1$ real, an equivalent form for (3.14) is ($f > 0$, say):

$$\begin{aligned} Ef^p\log f^p - EP_sf^p\log P_sf^p &\leq 2(1-e^{-2s})E|\nabla f^{\frac{p}{2}}|^2, \\ &= \frac{p^2(1-e^{-2s})}{2(p-1)}E\nabla f\cdot\nabla f^{p-1}, \quad s \geq 0. \end{aligned}$$

(iv) It is also clear from the form of the expansions (2.5) and (3.6) that the extremal functions, i.e., those for which equality in (2.1) and (3.15) holds, are restrictions to $[0,s]$ of the corresponding extremal function for $s=+\infty$, i.e., polynomials of proper order and exponential functions respectively.

As is well known, Gross' inequality is equivalent to the hypercontractivity of the Ornstein-Uhlenbeck semigroup. Similarly, we have the following which at $s=+\infty$ recovers the classical equivalence (below f satisfies the hypotheses of Theorem 3.1).

Corollary 3.4 *Let $1<p<+\infty$, let $p\le q\le 1+(p-1)e^{2t}$, $0\le t<+\infty$, and let $0\le s\le+\infty$. Then, the following two conditions are equivalent*

$$Ef^2\log f^2-EP_sf^2\log P_sf^2\le 2(1-e^{-2s})E|\nabla f|^2. \tag{3.18}$$

$$(E|P_tf|^q)^{\frac{1}{q}}\le e^{\int_0^t C(u,s,f)du}(E|f|^p)^{\frac{1}{p}}, \tag{3.19}$$

where $C(u,s,f)\le 0$ is given by (3.21) below.

Proof The proof follows the original proof of Gross (see [Ba]). To prove that (3.18) implies (3.19), let $\phi(t)=\log\left(E(P_tf)^{q(t)}\right)^{\frac{1}{q(t)}}$, where $q(t)\ (=q)=1+(p-1)e^{2t}$. Then, under the hypotheses on f,

$$\frac{d\phi(t)}{dt}=\frac{q'}{q^2E(P_tf)^q}\left(E(P_tf)^q\log(P_tf)^q-E(P_tf)^q\log E(P_tf)^q-\frac{q^2}{q'}E\nabla P_tf\cdot\nabla(P_tf)^{q-1}\right) \tag{3.20}$$

Now, since $\frac{dq}{dt}\ (=q')=2(q-1)$, and using Remark 3.3 (iii), it follows that

$$\frac{d\phi(t)}{dt}\le C(t,s,f),$$

where

$$C(t,s,f)=$$

$$\frac{2(q-1)}{q^2E(P_tf)^q}\left(EP_s(P_tf)^q\log P_s(P_tf)^q-E(P_tf)^q\log E(P_tf)^q-\frac{q^2e^{-2s}}{2(q-1)}E\nabla P_tf\cdot\nabla(P_tf)^{q-1}\right) \tag{3.21}$$

Since $\log\phi(0)=(E|f|^p)^{\frac{1}{p}}$, (3.19) now follows from (3.21). It is also clear by applying, to $(P_tf)^q$, (3.15) and (3.17) (both for $s=+\infty$) that the quantity (3.21) is non-positive, and in fact null at $s=+\infty$. To prove the converse implication, it is enough to set $p=2, q=1+e^{2t}$, to take the log in (3.19) and to differentiate at $t=0$.

Once more (thanks to the commutation property) the above results have versions on the two-point space, and for the Poisson kernel on the sphere. However, on Poisson space hypercontractivity and logarithmic Sobolev inequalities both fail as shown in Surgailis [S] and Bakry and Émery [BaE]. It is also probable that the results presented above have more abstract and/or general versions (for diffusions, for the heat semigroup on a compact Riemannian manifold) using the Γ_2 "philosophy" or the approach put forward in [L]. We must however, confess ignorance on that matter.

It is not clear (at least to the author of these lines) what (3.19), where C there is f-dependent will bring to the usual hypercontractive estimate. One way to test its usefulness might be to first try to estimate it in particular cases, e.g., on two-point space (where $P_t f = e^{-t} f$) or on the mth Wiener chaos (where $P_t I_m(f_m) = e^{-mt} I_m(f_m)$). Then, to compare these estimates to the known results (best constants in Khinchin inequality and equivalence of norms on the mth Wiener chaos). Another way to test (3.19) might be to see if its reinterpretation in terms of Riesz potentials (as done in Meyer [M] for classical hypercontractivity) brings some useful consequences.

To finish this note and to justify the claim made after the proof of Theorem 2.4, we state the following results on Wiener space. The results apply to "good" smooth functionals $F > 0$ and can be proved by combining the methods of Theorem 2.4 (with its notations) and the above ones.

$$\begin{aligned} 2EF\log F \quad - \quad & 2EP_sF\log P_sF = \sum_{k=1}^{n}(1-e^{-2s})^k\frac{(-1)^{k+1}}{k!}E < F, Q_k(\widetilde{\Gamma})\log F >_k \\ + \quad & \frac{(-1)^n}{n!}\int_0^s 2e^{2nt}(e^{-2t}-e^{-2s})^n E < P_tF, Q_{n+1}(\widetilde{\Gamma})\log P_tF >_{n+1} dt. \end{aligned} \quad (3.22)$$

Furthermore, the following two results are true and equivalent ($C(u,s,F)$ below is given as in (3.21), replacing however f there by F, and the term involving the gradient by $E < \nabla P_tF, \nabla (P_tF)^{q-1} >_1$).

$$EF^2\log F^2 - EP_sF^2\log P_sF^2 \le 2(1-e^{-2s})E||\nabla F||_1^2. \quad (3.23)$$

$$(E|P_tF|^q)^{\frac{1}{q}} \le e^{\int_0^t C(u,s,F)du}(E|F|^p)^{\frac{1}{p}}. \quad (3.24)$$

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